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Ergodic Theory of the Mixmaster Universe in Higher Space-Time Dimensions. II

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The topological dynamics of the mixmaster models in space-time dimension $d+1$ are investigated. We use a new parametrization to reduce the mixmaster map to a translation combined with an appropriate isometry or a dilating inversion. For $d \leqslant 9$, we show that the mixmaster map is ergodic and topologically mixing. For $d \geqslant 10$, the mixmaster map reduces to the identity after a finite number of iterations, except for a set of initial data with zero Lebesgue measure.

KEY WORDS: Mixmaster model; ergodic theory; chaos; Einstein equations; cosmological singularity; Kaluza–Klein cosmology.

1. INTRODUCTION

Recent efforts to develop a unified theory of fundamental interactions have revived interest in gravitational theories in higher space-time dimensions. Contrasting with our $(3+1)$-dimensional space-time, one of the basic properties of the solutions investigated is that they should be anisotropic, to account for the asymmetry between the ordinary three spatial dimensions and the extra spatial dimensions. A second motivation for considering anisotropic space-time models is that they have a high degree of generality and may therefore provide insight into fundamental properties of the space-time models. We have thus undertaken the analysis[12,13] of the anisotropic solutions of the Einstein equations in $(d+1)$ space-time dimensions with the methods of topological dynamics. This analysis was prompted by the interesting dependence of the qualitative behavior of these cosmological models on the number of spatial dimensions,[10,11] and the results presented here were obtained in collaboration with M. Henneaux.

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Anisotropic solutions of the Einstein equations can be described in terms of the generalized Kasner metric

$$ds^2 = -dt^2 + \sum_{i=1}^{d} t^{2p_i(x)}(l^i_j \, dx^j)^2 \tag{1.1}$$

where $\det(l^i_j) \neq 0$; the exponents $p_i(x)$ belong to the "Kasner sphere"

$$\sum_{i=1}^{d} p_i^2 = \sum_{i=1}^{d} p_i = 1 \tag{1.2}$$

and are assumed to be in increasing order

$$p_1 \leqslant p_2 \leqslant \cdots \leqslant p_d \tag{1.3}$$

As at least p_1 is negative, while p_{d-1} and p_d are positive, this universe model is expanding in at least one direction and contracting in at least two directions when it approaches the initial singularity. The time evolution of the exponents p_i depends on the matrices $(l^i_j(x))$; they are constant for the homogeneous vacuum Kasner solution (with $l^i_j = \delta^i_j$).

One interesting feature of the $(3+1)$-dimensional solution is that the metric (1.1) with inhomogeneous (space-dependent) l^i_j behaves qualitatively in the same way as the homogeneous solutions[2–7,17–20]; for "almost all" initial conditions, the exponents p_i remain almost constant over a time interval and then suddenly change to values that are given by a single rational map. At such a "collision" the dilating and contracting directions also exchange their roles. This process of smooth evolution interrupted by brief "collisions" repeats indefinitely and becomes more violent as one approaches the singularity. The collision map $(p_i \to p'_i)$ possesses strong chaotic properties and lends itself to a statistical description.[2,3,8,17,18]

In $(d+1)$ dimensions $(d \geqslant 4)$, various authors have found that the anisotropic homogeneous vacuum solutions of the Einstein equations do not exhibit similar infinite sequences of oscillations.[5,14–16,21] However, the presence of small inhomogeneities induces collisions analogous to the three-dimensional case:

$$p'_i = \mathrm{Ord}(q_i) \tag{1.4}$$

where the ordering operator Ord, ensuring (1.3), acts on the sequence q:

$$q_1 = (p_1 - \alpha)/(1 + \alpha) \tag{1.5a}$$

$$q_i = p_i/(1 + \alpha) \qquad (2 \leqslant i \leqslant d-2) \tag{1.5b}$$

$$q_{d-1} = (p_{d-1} + \alpha)/(1 + \alpha) \tag{1.5c}$$

$$q_d = (p_d + \alpha)/(1 + \alpha) \tag{1.5d}$$

provided that the quantity (denoted $\alpha_{1,d-1,d}$ in Refs. 10–12)

$$\alpha = 1 + p_1 - p_{d-1} - p_d \tag{1.6}$$

is negative. If α is positive, then no collision occurs, i.e.,

$$p_i' = p_i, \qquad 1 \leqslant i \leqslant d \tag{1.7}$$

and the metric (1.1) smoothly approaches the singularity. This condition, $\alpha \geqslant 0$, cannot be satisfied if $d \leqslant 9$, except at isolated points, but for $d \geqslant 10$ the condition $\alpha > 0$ does define an open nonempty set in the Kasner sphere for which the Kasner regime is stable. Further investigation suggested the following conjectures[10,11]:

1. The mixmaster map (1.4)–(1.5) is chaotic for $d \leqslant 9$, as it is for $d = 3$.
2. For $d \geqslant 10$, almost any initial condition ultimately reaches the Kasner stability region $\alpha > 0$, in which the oscillatory behavior ceases.

It is well known[2,3,8,9,17,18] that, for $d = 3$, the map (1.4)–(1.6) is related to the continued-fraction transformation of Gauss, which has good ergodic properties. In a first step toward a proof of the conjectures, Elskens and Henneaux[12] proved that the mixmaster map is topologically mixing for $d = 4$ and introduced a set of variables in which the map decomposes into an isometry and a dilatation. The present work relies on this change of variables to complete the proof of the two conjectures.

This paper is organized as follows. In Section 2, we formulate the map (1.4)–(1.6) in the new variables. In Section 3, we introduce the basic partition of the ordered sector of the Kasner sphere and state our theorems; the proofs are left for the appendices. Section 4 is devoted to short comments.

2. THE MIXMASTER MAP IN REDUCED VARIABLES

The Kasner conditions $\sum_{i=1}^{d} p_i^2 = \sum_{i=1}^{d} p_i = 1$ define a $(d-2)$-dimensional sphere, which is in one-to-one correspondence with the space $\mathbb{R}^{d-2} \cup \{\infty\}$, in which the coordinates (u_i), $1 \leqslant i \leqslant d-2$, are defined as follows:

1. If $p_d = 1$, then $p_i = 0$ $(i < d)$ by the Kasner conditions, and we write $u = \infty$.
2. If $p_d < 1$, then we let

$$u_i = p_i/(1 - p_d) \tag{2.1}$$

The mapping $(p_i) \to (u_i)$ amounts to a stereographic projection of the Kasner sphere from the pole $(0, 0, ..., 1)$ onto the plane $(x_d = 2, \sum_{i=1}^{d} x_i = 1)$. As the pole $(0, ..., 0, 1)$ is invariant under the map (1.4)–(1.5) it is harmless to reject it at infinity in the new representation.

If we introduce the additional variables

$$u_{d-1} = p_{d-1}/(1 - p_d), \qquad u_d = p_d/(1 - p_d) \tag{2.2}$$

it is easily seen that

$$\begin{gathered} \sum_{i=1}^{d-1} u_i = 1 \\ u_d = \frac{1}{2} \sum_{i=1}^{d-1} u_i^2 - \frac{1}{2} \\ p_i = u_i/(1 + u_d) \qquad (1 \leqslant i \leqslant d) \end{gathered} \tag{2.3}$$

Finally, we also define the quadratic form

$$f(u) = \frac{1}{2} \sum_{i=1}^{d-2} u_i^2 + \frac{1}{2} \left(\sum_{i=1}^{d-2} u_i \right)^2 = u_d - u_{d-1} + 1 \tag{2.4}$$

We call the variables (u_i), $1 \leqslant i \leqslant d$, the reduced variables corresponding to the exponents (p_i). The inequalities $p_1 \leqslant \cdots \leqslant p_d$ define a subset A of $\mathbb{R}^{d-2}$:

$$u_1 \leqslant u_2 \leqslant \cdots \leqslant u_{d-2} \leqslant 1 - \sum_{i=1}^{d-2} u_i, \qquad f(u) \geqslant 1 \quad (\Leftrightarrow u_d \geqslant u_{d-1}) \tag{2.5}$$

and we denote by $\bar{A}$ its closure $A \cup \{\infty\}$. For $d = 3$, this construction reduces to the parametrization of the Kasner circle S_1 given by Belinskii *et al.*[6,7] (with their $u = -u_1$). We endow A with the distance $d(u, u')$:

$$d(u, u')^2 = \frac{1}{2} \sum_{i=1}^{d-1} (u_i - u_i')^2 = \frac{1}{2} \frac{\sum_{i=1}^{d} (p_i - p_i')^2}{(1 - p_d)(1 - p_d')} \tag{2.6}$$

and write $d(u, \infty) = \infty$.

Any point on the Kasner sphere can be brought to the ordering (2.5) by permuting some exponents (p_i). The corresponding transformations of the u_i have been discussed in Ref. 12; they are combinations of the following transformations:

1. To permute any of the $(d-1)$ first exponents $(p_1, ..., p_{d-1})$, one performs the same permutation on the reduced variables; this transformation preserves the distance (2.6) and its eigenvalues have unit modulus.

2. To exchange p_{d-1} and p_d, one performs the inversion with respect to the quadric $f(u) = 1$:

$$u_i' = u_i/f, \qquad 1 \leqslant i \leqslant d-2 \tag{2.7}$$

so that $u_{d-1}' = u_d/f$ and $u_d' = u_{d-1}/f$; this transformation does not preserve the distance (2.6), and its eigenvalues are $\pm 1/f$.

Thus, if one defines the exponents (p_i) by placing the exponents (q_i) in increasing order, two cases occur:

1. If q_d is the largest of all q_i, the corresponding transformation for the reduced variables is an isometry (a permutation).
2. If q_d is not the largest of all q_i, one first orders $(q_1 \cdots q_{d-1})$ by an isometry; then one performs the inversion (2.7), which is a dilatation, since $f < 1$ for these exponents; and another isometry brings the last exponent to its place; the resulting transformation has eigenvalues with modulus $1/f > 1$.

With these reduced variables, the mapping (1.5) appears as a translation parallel to the u_1 axis

$$T: \mathbb{R}^{d-2} \to \mathbb{R}^{d-2}: u \to u' = Tu = (u_1 + 1, u_2, \ldots, u_{d-2}) \tag{2.8}$$

and we define, according to (2.3),

$$u_{d-1}' = u_{d-1} - 1, \qquad u_d' = u_1 + f(u) \tag{2.9}$$

To be complete, we specify that $T\infty = \infty$.

This transformation T does not necessarily preserve the ordering of u. If $u_1' > u_2$ or $u_{d-2}' > u_{d-1}'$ or $u_{d-1}' > u_d'$, one needs to perform a permutation and/or an inversion to bring Tu into $\bar{A}$. These reorderings, which are considered in the next section in more detail, are the source of the various dynamical behaviors of the mixmaster map.

Remark. Our definition (2.8)–(2.9) of the mixmaster map in the reduced variables differs from the definition of Section 1 for points such that $\alpha > 0$ (where the original map reduces to the identity). This difference is motivated by the desire to use the same expression for Tu for any $u \in A$; it is harmless because $T_A^2 u = u$ if $\alpha > 0$. Note that

$$(1 + u_d)\,\alpha = 1 + u_1 - u_{d-1} \tag{2.10}$$

3. THE PARTITIONS P_d AND P_d'

The mixmaster map (1.4)–(1.5) is thus written as $T_A: \bar{A} \to \bar{A}$: $u \to STu = x$ in terms of the reduced variables, where S is the appropriate reordering transformation. The actual form of S depends on the point u considered, and the various orderings in which $u' = Tu$ may come out define a partition P_d of $\bar{A}$. For an arbitrary $u \in A$, the ordering of u' is constrained as follows:

$$u'_2 = u_2 \leqslant u'_3 \leqslant \cdots \leqslant u'_{d-2}, \qquad u'_d = u_1 + f(u) \geqslant u'_1 = u_1 + 1$$

Thus, the complete order of the variables u'_i is fixed by the positions of u'_1, u'_{d-1}, and u'_d in the sequence; the partition P_d of $\bar{A}$ may thus be written

$$P_d = \{[i, j, k]\}, \qquad 1 \leqslant i < k \leqslant d, \quad 1 \leqslant j \leqslant d \tag{3.1}$$

with $j \neq k$, $j \neq i$; the cell $[i, j, k]$ is

$$[i, j, k] = \{u \in A \colon p'_1 = q_i,\ p'_{d-1} = q_j,\ p'_d = q_k\} \tag{3.2}$$

where (p'_i) and (q_i) are the exponents corresponding to (u'_i) and (x_i), respectively.[3] Some cells defined by (3.2) may be empty for some values of d (especially if $d \leqslant 8$) or reduce to isolated points, but this will not alter our discussion. As our definition implies that all cells are closed, some points of $\bar{A}$ belong to more than one cell, but they form a set of codimension one (or higher) with total measure zero.

Two regions in A present a major interest:

1. The *inversion region* $B = \operatorname{int} \bar{B}$, where

$$\bar{B} = \bigcup_{k<d} [i, j, k] \tag{3.3}$$

is the set of points of A for which the reordering of $u' = Tu$ involves the dilating inversion (2.7).

2. The *Kasner stability region* $K = \operatorname{int} \bar{K}$, where $\alpha > 0$; it is a subset of the *extended stability region*

$$\bar{G} = [d-1, 1, d] \tag{3.4}$$

on which T_A^2 reduces to the identity.

The inversion region B is nonempty for all values of d. Both $\bar{K}$ and $\bar{G}$ are empty for $d \leqslant 8$ and reduce to a single point $\{c\}$ for $d = 9$; for $d \geqslant 10$, they have a nonempty interior, which plays a fundamental role in the dynamics.

[3] We use the exponents instead of the reduced variable because the reordering S is then written as a simple permutation. See also Appendix A.

In Appendix A, we prove the following result:

Theorem 1. For any $d \geqslant 3$ and $[i, j, k] \in P_d$, either $T_A^5[i, j, k] = \bar{A}$ or $T_A[i, j, k] = \bar{K}$ or $T_A[i, j, k] = \bar{G}$.

Of course, for $d \leqslant 8$, only the first case occurs; and for $d = 9$, the first case occurs for all cells with a nonzero measure. It was proved in Ref. 12 that the partition P_d is generating for $d \leqslant 9$, and that T_A^2 is dilating along all directions in B (except near c and its preimage c' by T for $d = 9$). This leads to the following result:

Theorem 2. For $d \leqslant 9$, T_A is topologically mixing, it has positive topological entropy, and it is ergodic.

The proof is given in Appendix B.

For $d \geqslant 10$, the fact that $\bar{K}$ has a nonempty interior drastically changes the dynamics. We show in Appendix C that the same properties hold for T_A outside $\bar{G}$ as hold there for $d = 9$. Moreover, by considering in Appendix D an adequate modification of T_A (mapping $\bar{G}$ onto $\bar{A}$ again), we also prove the following result:

Theorem 3. If $d \geqslant 10$, the set of points that T_A maps into $\bar{K}$ is dense in $\bar{A}$, and its complement has Lebesgue measure zero.

There is no contradiction between this result and Theorem 1: Theorem 1 is concerned with the image of cells by a finite number of iterations, whereas Theorems 2 and 3 deal with the asymptotic evolution of the points. In fact, Theorem 1 ensures that, for any $m \geqslant 0$, there is a subset $U \subset \bar{A}$ such that $T_A^m U = \bar{A}$ even if $d \geqslant 10$.

We have thus completed the proof of the conjectures proposed by Demaret *et al.*[10,11]

4. CONCLUDING REMARKS

From the physical viewpoint (see also Ref. 13), these results show that the presence of a small spatial inhomogeneity can completely change the qualitative behavior of some simple solutions to the Einstein equations. But other classes of cosmological models may exhibit completely different behaviors. We shall thus confine ourselves to short topical remarks.

First, the role of the (unstable) fixed points and periodic orbits of the map for $d \leqslant 9$ calls for further study. Eras starting with exponents near the pole $(0,..., 0, 1)$ may be arbitrarily long: for $d = 3$, this makes the Liapunov exponents and K-entropy of T_A vanish. But the return map T_B in the inversion domain should be genuinely chaotic. Note also that, like $\partial T_A u/\partial u$, the derivative $\partial T_B u/\partial u$ is the product of an isometry and a homothety (with ratio $\geqslant 1$) and thus has all its Liapunov exponents equal.

Besides, the application of ergodic theory to a cosmological model suggests that a statistical mechanical description of the gravitational field may be natural near a singularity for some broad class of space-times; such a description would be complementary to the exact solutions (which often call on special symmetries) and certainly deserves further attention.

APPENDIX A

In this Appendix, we establish the conditions a point $x \in A$ must satisfy in order to have a preimage $u \in A$ within any cell $[i, j, k]$ of the partition P_d; let $u' = Tu$. Let p, p', and q denote the exponents corresponding to u, u', and x, respectively, through (2.1)–(2.2). As $x = T_A u$, we have

$$\begin{aligned} p'_1 &= (u_1+1)/(u_1+f_u+1), & q_1 &= x_1/(x_d+1) \\ p'_l &= u_l/(u_1+f_u+1), & q_l &= x_l/(x_d+1) \qquad (2 \leqslant l \leqslant d-2) \\ p'_{d-1} &= (u_{d-1}+1)/(u_1+f_u+1), & q_{d-1} &= x_{d-1}/(x_d+1) \\ p'_d &= (u_1+f_u)/(u_1+f_u+1), & q_d &= x_d/(x_d+1) \end{aligned} \tag{A.1}$$

and

$$q = \mathrm{Ord}(p') \tag{A.2}$$

We write $f_u = f(u)$ and $f_x = f(x)$.

Of course, $x \in A$ implies $x_1 \leqslant \cdots \leqslant x_{d-1}$ and $f_x \geqslant 1$. If x is the image of $u \in [i, j, k]$ by T_A, then

$$p'_1 = q_i, \qquad p'_{d-1} = q_j, \qquad p'_d = q_k \tag{A.3}$$

Introducing

$$\beta_k = 1 + x_d - x_k \geqslant 1 \tag{A.4}$$

we recast the system (A.1)–(A.2) in the form

$$(x_1,\ldots, x_d)/\beta_k = \mathrm{Ord}(u'_1,\ldots, u'_d) \tag{A.5}$$

Knowing from (A.3) the actual order of u', we may invert the reordering transformation and solve (A.5) for u as a function of x.

The solutions u so obtained may fall outside the ordered region A: a point x need not have preimages by T_A in all cells of P_d. We must therefore ensure that $f(u) \geqslant 1$ and $u_1 \leqslant \cdots \leqslant u_{d-1}$. The first condition is satisfied provided that $p'_d \geqslant p'_1$, i.e., $k \geqslant i$. The conditions $u_2 \leqslant \cdots \leqslant u_{d-2}$ are automatically satisfied (see Section 2). The only restrictions requiring special verification are $u_1 \leqslant u_2$ and $u_{d-2} \leqslant u_{d-1}$; in view of (A.1)–(A.3), these conditions read:

1. If $i \geqslant 2$:

$$u_1 = u'_1 - 1 = (x_i/\beta_k) - 1 \leqslant u_2 = u'_2$$

Depending on whether $j = 1$ or $j > 1$, we have $u'_2 = x_2/\beta_k$ or $u'_2 = x_1/\beta_k$, respectively, which yields a necessary and sufficient condition on x.

2. If $j \leqslant d-2$:

$$u_{d-1} = (x_j/\beta_k) + 1 \geqslant u_{d-2} = u'_{d-2}$$

and $u'_{d-2} = x_{d-2}/\beta_k$ or $u'_{d-2} = x_{d-1}/\beta_k$ or $u'_{d-2} = x_d/\beta_k$, depending on the values of i and k.

The resulting conditions are further detailed in Table I.

The cells $[i, j, d]$ with $i > j$ deserve special attention. We consider first the cells $[d-1, j, d]$, $1 \leqslant j \leqslant d-2$, which are nonempty[4] only for $d \geqslant 9$ (and reduce to isolated points for $d = 9$). If $u \in [d-1, 1, d]$, then $T_A u \in [d-1, 1, d]$ and $T_A^2 u = u$. And if $u \in [d-1, j, d]$ with $2 \leqslant j \leqslant d-2$, then $x_1 = u'_2 = u_2$, so that (for $x' = Tx$) $x'_{d-1} = u_1 \leqslant u_2 \leqslant x_2$, while $x'_1 = u_2 + 1 \geqslant u_1 + 1 \geqslant x_{d-2}$: thus, $x \in [d-1, 1, d]$. Second, if $u \in [d-2, j, d]$ with $1 \leqslant j \leqslant d-3$, then $x \in [d-1, j', d]$ with $j' \leqslant j$, so that $T_A^2 u \in [d-1, 1, d]$. Similarly, if $u \in [d-3, 2, d]$, then $T_A^2 u \in [d-1, 1, d]$ also. Therefore, in all these cases, $T_A^n u \in [d-1, 1, d]$ for any $n \geqslant 2$, i.e., the orbit of u is trapped in the extended stability region.

Table I may also be interpreted in analogy with the compatibility matrices of topological Markov chains.[1,12] For each cell $[i, j, k]$ of P_d, our conditions determine a part of A covered by its image $T_A[i, j, k]$; it is then easy to find one or more cells of P'_d covered by this image. If this image reduced exactly to a union of cells of P_d (resp. P'_d), the partition P_d (resp. P'_d) would be Markovian, but this only occurs for $d = 3$ and $d = 4$. Note that Table I does not define a "compatibility" matrix for P_d (since some cells are mapped onto $\bar{K}$ rather than over the whole cell $[d-1, 1, d]$ of P_d) and that only the cells belonging to B ($k \leqslant d-2$) have their boundaries mapped on surfaces that are not always cell boundaries themselves.

For each cell $[i, j, k]$ of P'_d, the determination of cells completely covered by its image then enables us to deduce also a maximum number n

[4] For some values of d, Table I includes empty cells (e.g., $[1, 3, 2]$ for $d = 3$). This has no impact on our conclusions, because the image of an empty set is empty.

Table I[a]

u'_1	u'_{d-1}	u'_d	Conditions for $x = T_A u$		Image cells	Number n
1	$d-1$	d	—	—	$\bar{A}$	1
i	$d-1$	d	$x_i \leqslant x_1 + 1$	—	$[i, d, d-1]$	3
1	j	d	—	$x_j \geqslant x_{d-1} - 1$	$[1, j+1, j]$	3
$i <$	j	d	$x_i \leqslant x_1 + 1$	$x_j \geqslant x_{d-1} - 1$	$[i, j, d-1]$	5
$i >$	j	d	$x_i \leqslant x_1 + 1$	$x_j \geqslant x_{d-1} - 1$	$[i+1, j-1, d-1]$	5
$d-3$	2	d	$x_{d-3} \leqslant x_1 + 1$	$x_2 \geqslant x_{d-1} - 1$	$[d-2, 1, d-1]$	—
$d-2$	j	d	$x_{d-2} \leqslant x_1 + 1$	$x_j \geqslant x_{d-1} - 1$	$[d-1, j, d]$	—
$d-1$	j	d	$x_{d-1} \leqslant x_1 + 1$		$\bar{K}$	—
i	1	d	$x_{d-1} \leqslant x_1 + 1$		$\bar{K}$	—
$d-1$	1	d	$x_{d-2} \leqslant x_1 + 1$	$x_2 \geqslant x_{d-1} - 1$	$[d-1, 1, d]$	—
1	d	$d-1$	—	—	$\bar{A}$	1
i	d	$d-1$	$x_i \leqslant x_1 + f_x$	—	$[1, d-1, d]$	2
1	$d-2$	$d-1$	—	$x_{d-2} \geqslant x_{d-1} - 1$	$[1, d-1, d-2]$	2
i	$d-2$	$d-1$	$x_i \leqslant x_1 + f_x$	$x_{d-2} \geqslant x_{d-1} - 1$	$[1, d-1, d-2]$	2
1	j	$d-1$	—	$x_j \geqslant x_{d-1} - 1$	$[1, j, d-2]$	4
i	j	$d-1$	$x_i \leqslant x_1 + f_x$	$x_j \geqslant x_{d-1} - 1$	$[1, j, d]$	4
2	1	$d-1$	—	$x_1 \geqslant x_{d-1} - 1$	$\bar{K}$	—
i	1	$d-1$	$x_1 \geqslant x_{d-1} - 1$		$\bar{K}$	—
1	d	k	—	—	$\bar{A}$	1
i	d	k	$x_i \leqslant x_1 + \beta_k$	—	$[1, d, d-1]$	2
1	$d-1$	k	—	—	$\bar{A}$	1
i	$d-1$	k	$x_i \leqslant x_1 + \beta_k$	—	$[1, d, d-1]$	2
1	j	k	—	$x_j \geqslant x_k - 1$	$[k, d, d-1]$	3
i	j	k	$x_i \leqslant x_1 + \beta_k$	$x_j \geqslant x_k - 1$	$[k, d, d-1]$	3
2	1	k	—	$x_1 \geqslant x_k - 1$	$[k, d, d-1]$	3
i	1	k	$x_k \leqslant x_1 + 1$		$[k, d, d-1]$	3

[a] The first three indices identify each cell of the partition P_d; when unspecified, they take values $2 \leqslant i \leqslant d-2$, $2 \leqslant j \leqslant d-2$, and $2 \leqslant k \leqslant d-2$ with $k \geqslant i+1$. The inequalities listed are necessary and sufficient conditions on x for finding $u \in A$; the "image cells" list some cells of P'_d satisfying these conditions. The last column lists n such that $T_A^n[i, j, k] = \bar{A}$. The partition P'_d differs from P_d only by subdividing $[d-1, 1, d]$ into $\bar{K}$ and $\bar{K}'$.

of iterations such that $T_A^n[i, j, k] = \bar{A}$ (except for the trapped cells mapped only on $\bar{K}$ or $\bar{G}$), which completes the proof of Theorem 1.

It also appears from Table I that at least $d-1$ cells are mapped over the whole $\bar{A}$, and that (as noted in Refs. 10 and 11) the image of any cell also covers $\bar{K}$.

APPENDIX B. TOPOLOGICAL CHAOS AND ERGODICITY FOR $d \leqslant 9$

Starting from Theorem 1 and from the dilating property of the mapping T_A, we establish the strongly chaotic properties of T_A.

Let $u \in A$. For any $n \geqslant 0$, let $\Delta_n(u)$ be the cell of P_d to which $T_A^n u$ belongs. Since P_d is generating, the sequence $(\Delta_n(u))$, $0 \leqslant n < \infty$, uniquely determines u; in fact,

$$\lim_{n \to \infty} \Gamma_n(u) = \{u\} \tag{B.1}$$

where

$$\Gamma_n(u) = \bigcap_{k=0}^{n} T_A^{-k} \Delta_k(u) \tag{B.2}$$

The sets $\Gamma_n(u)$ are closed. Consider now any open set $U \ni u$. Because of (B.1), there must be an integer $m > 0$ such that $\Gamma_m(u) \subset U$; then, according to Theorem 1,

$$T_A^{m+5} U \supset T_A^{m-k+5} \Delta_k(u) = \bar{A} \tag{B.3}$$

As a result, for any open sets U, $V \subset A$, there is some $N > 0$ such that, $\forall n > N$, $T_A^n U \cap V \neq \varnothing$. The mixmaster map T_A is thus *topologically mixing.* It is also clear that T_A has positive topological entropy $h_T \geqslant (1/5) \ln N_d$, where N_d is the number of cells of P_d with a nonempty interior.[1]

Consider now a (Lebesgue) measurable set U', with nonzero measure, differing from U only by a null set. Then $T_A^{m+5} U'$ also differs from $\bar{A}$ only by a null set. Therefore, the only invariant measurable subsets of $\bar{A}$ are either null sets or have a null complement: T_A is *ergodic.*

In fact, the previous results are much stronger: if the mapping T_A admits an invariant measure absolutely continuous with respect to the Lebesgue measure, the map T_A must be *exact* with respect to this measure.[9,12] However, this invariant measure is not yet known for $4 \leqslant d \leqslant 9$.

APPENDIX C. THE MIXMASTER MAP OUTSIDE THE STABILITY REGION

In this Appendix, we extend the results of Ref. 12 to the case $d \geqslant 10$, where the Kasner stability region has a nonempty interior: we prove that any point $u \in A$ is eventually mapped[5] into either the inversion region B or the Kasner stability region $\bar{K}$, and that T_A^2 is uniformly expanding over B, except near the boundary of $\bar{K}$. Before embarking on the proofs, let us define for any $u \in A$ (with $u' = Tu$)

$$g(u) = \max(u'_{d-1}, u'_{d-2}) - u'_1 \tag{C.1}$$

[5] If a point u is mapped into the extended stability region $\bar{G}$, the next iteration of T_A maps it into $\bar{K}$.

By continuity, let $g(\infty) = +\infty$. Since $u'_1 \leqslant u'_d$ and $u'_2 \leqslant \cdots \leqslant u'_{d-2}$, we see that $u \in [d-1, j, d]$ for some $j \leqslant d-2$ if and only if $g(u) \leqslant 0$; in that case, $T_A u \in \bar{K}$.

We also introduce

$$\sigma(u) = d(u, \bar{K}) = \inf_{c \in \bar{K}} d(u, c) = d(u, \partial \bar{K}) \tag{C.2}$$

with the distance (2.6). Note that g is a continuous (piecewise linear) function of u, vanishing in A only for $u \in \partial \bar{K}$; thus, $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in [0, \varepsilon_0]$, $\exists \gamma > 0$ such that $g(u) > \gamma$ if $\sigma(u) > \varepsilon$.

Proposition. $\forall u \in A$, $\exists n \in \mathbb{N}$: $T_A^n u \in B \cup \bar{K}$.

Proof. Consider $u \in A$. In view of the previous remarks, we may assume that $g(T_A^n u) > 0$ for any $n > 0$, and that $u \notin B$. Let $x = T_A u$.

As $u \notin B$, $x_d = u'_d$ and

$$f(x) = x_d - x_{d-1} + 1 = f(u) - g(u) \tag{C.3}$$

Thus, $f(T_A^n u)$ decreases as n increases, unless $T_A^n u \in B \cup \bar{K}$ for some n. Therefore, if there is any $\gamma > 0$ such that, $\forall N > 0$, $\exists n > N$: $g(T_A^n u) > \gamma$, then there is some $m > 0$ such that $f(TT_A^m u) < 1$, and $T_A^m u \in B$. It is now sufficient to consider only the case where $\sigma(T_A^n u) \to 0$ as $n \to \infty$. Consider the cell $\Delta_n \in P'_d$ to which $T_A^n u$ belongs: for ε small enough, Δ_n must contain at least one point $c \in \partial \bar{K}$. Moreover, $\Delta_n \subset \overline{A \setminus (B \cup K)}$, since $T_A^n u \in B \cup K$. Then $T_A \Delta_n$ is a union of similar cells of P'_d (see Appendix A), and T_A^2 acts on $\Delta_n \cap T_A^{-1} \Delta_{n+1}$ as an isometry. As $c = T_A^2 c$ (since $c \in \bar{K}$), we find that $d(T_A^{n+2} u, c) = d(T_A^n u, c)$, and:

1. If the boundary $\partial \Delta_n \cap \partial \bar{K}$ is $(d-3)$-dimensional, then $\sigma(T_A^{n+2} u) = \sigma(T_A^n u)$, and $\sigma(T_A^m u)$ would not converge to zero for $m \to \infty$ (which contradicts our previous assumption).
2. If the boundary $\partial \Delta_n \cap \partial K$ is r-dimensional ($r < d-3$), $T_A^m u$ cannot approach closer to this part of the boundary $\partial \bar{K}$: if $\sigma(T_A^n u) \to 0$ as $n \to \infty$, the point $T_A^m u$ must approach $\partial \bar{K}$ through another cell, and ultimately $\partial \Delta_m \cap \partial \bar{K}$ through another cell, and ultimately $\partial \Delta_m \cap \partial \bar{K}$ must be $(d-3)$-dimensional.

Thus, we arrive to a contradiction, and the proof is complete. ∎

The preceding argument also indicates that T_A^2 is uniformly expanding over B, except near ∂K: it suffices to reproduce the proof of Ref. 12 (Theorem 2), replacing only the case $(d = 9,\ u = c)$ by $(d \geqslant 9,\ u \in \bar{K})$.

APPENDIX D. A VARIANT OF THE MIXMASTER MAP

To complete our proofs, we introduce an auxiliary mapping $\tilde{T}_A$, and a subset $C \subset \bar{K}$ so that:

1. $\tilde{T}_A$ is continuous over $\bar{A}$
2. $\tilde{T}_A u = T_A u$ for any $u \in \bar{A} \backslash K$
3. $\tilde{T}_A \bar{C} = \bar{A}$, and $\tilde{T}_A$ is one-to-one and differentiable on $\bar{C}$
4. $\tilde{T}_A(\bar{K} \backslash \text{int } C) = \bar{A} \backslash \text{int } K'$, and $\tilde{T}_A$ is one-to-one and differentiable on $\bar{K} \backslash \text{int } C$
5. $\tilde{T}_A(\bar{K} \backslash \partial A) \subset \partial A$

We then define the partition $\tilde{P}_d$ by replacing in P_d the cell $[d-1, 1, d]$ by three cells $\overline{K'} = \overline{G \backslash K}$, $\overline{C'} = \overline{K \backslash C}$, and $\bar{C}$. Any cell of $\tilde{P}_d$ is thus mapped onto $\bar{A}$ a finite number of iterations of $\tilde{T}_A$.

We further require that, for any $u \in K$, the Jacobian matrix $\partial T_A u/\partial u$ be dilating by a factor $\beta(u) \geqslant 1$ in all directions, with $\beta(u) = 1$ only if $u \in \partial \bar{K}$. With this definition, $\tilde{P}_d$ is generating for $\tilde{T}_A$: if two points u, $u' \in \bar{A}$ belong to the same cell of $\tilde{P}_d$, the mapping $\tilde{T}_A$ never reduces their distance; unless their trajectories end up in $\partial \bar{K} \cup \partial \bar{K}'$, the distance $d(T_A^n u, T_A^n u')$ eventually exceeds the diameter of $B \cup \bar{K}$, and the two trajectories must fall in distinct cells of $\tilde{P}_d$. As to the points ending on cell boundaries, they form a subset of A with an empty interior and measure zero ($\partial \bar{K}$ has dimension $d-3$, and so does $\tilde{T}_A^{-n}\, \partial \bar{K}$ for any n).

The fact that any cell of $\tilde{P}_d$ is eventually mapped onto $\bar{A}$, and that $\tilde{P}_d$ is generating, enables one to prove that, for any neighborhood $V(u)$ of any point $u \in \bar{A}$, $\exists N > 0$ such that $\tilde{T}_A^N V(u) = A$. Therefore, $\tilde{T}_A$ shares the properties that hold for T_A in $d \leqslant 9$ (see Appendix B); in particular, it is topologically mixing and it is ergodic.

Subsequently, for any point $u \in \bar{A}$, there is a point $u' \in A$ arbitrarily close to u, and such that $\tilde{T}_A^n u \in \bar{K}$ for some n: the trajectory of u' under the mixmaster map T_A ends in the Kasner stability region. Thus, the set

$$Q = \{u \colon \lim_{n \to \infty} T_A^n u \in K\}$$

is dense in $\bar{A}$. Moreover, its complement $\bar{A} \backslash Q$ has measure zero; indeed, $\tilde{T}_A(\bar{A} \backslash Q) = T_A(\bar{A} \backslash Q) = \bar{A} \backslash Q$: if $\bar{A} \backslash Q$ had a positive measure, $\tilde{T}_A$ would not be ergodic.

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REFERENCES

1. V. M. Alekseev and M. V. Yakobson, *Phys. Rep.* **75**:287–325 (1981).
2. J. D. Barrow, *Phys. Rep.* **85**:1–49 (1982).
3. J. D. Barrow, in *Classical General Relativity*, W. B. Bonner, J. N. Islam, and M. A. H. MacCallum, eds. (Cambridge University Press, Cambridge, 1984), pp. 26–41.
4. J. D. Barrow and F. J. Tipler, *Phys. Rep.* **56**:371–402 (1979).
5. J. D. Barrow and J. Stein-Schabes, *Phys. Rev. D* **32**:1595–1596 (1985).
6. V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, *Adv. Phys.* **19**:525–573 (1970).
7. V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, *Phys. Lett.* **77A**:214–216 (1980).
8. D. F. Chernoff and J. D. Barrow, *Phys. Rev. Lett.* **50**:134–137 (1983).
9. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer, Berlin, 1982).
10. J. Demaret, M. Henneaux, and P. Spindel, *Phys. Lett.* **164B**:27–30 (1985).
11. J. Demaret, J. L. Hanquin, M. Henneaux, P. Spindel, and A. Taormina, *Phys. Lett.* **175B**:129–132 (1986).
12. Y. Elskens and M. Henneaux, Ergodic theory of the mixmaster model in higher space-time dimensions, *Nucl. Phys. B*, in press.
13. Y. Elskens and M. Henneaux, Chaos in Kaluza–Klein models, *Class. Quant. Gravity*, in press.
14. T. Furusawa and A. Hosaya, *Prog. Theor. Phys.* **73**:467–475 (1985).
15. P. Halpern, *Phys. Rev. D* **33**:354–362 (1986).
16. H. Ishihara, *Prog. Theor. Phys.* **74**:490–500 (1985).
17. I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, L. M. Shchur, and Ya. G. Sinai, *J. Stat. Phys.* **38**:97–114 (1985).
18. E. M. Lifshitz, I. M. Lifshitz, and I. M. Khalatnikov, *Sov. Phys. JETP* **32**:173–180 (1971).
19. C. W. Misner, *Phys. Rev. Lett.* **22**:1071–1074 (1969).
20. M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, 1974).
21. A. Tomimatsu and H. Ishihara, *Gen. Rel. Grav.* **18**:161–171 (1986).